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A Ground State Phase Transition in a 1^{o} Spin System: An Analog of ϕ^4 Field Theory

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ABSTRACT

INTRODUCTION

Understanding the properties of local field theories which undergo a spontaneous breaking of an internal symmetry has become one of the most significant problems in elementary particle physics. Such a mechanism has been invoked for some time to explain the structure of weak interactions.[1] With the dawning realization that the strong interactions may be understood in terms of the phase or symmetry properties of the ground state of a local gauge field theory[2] the problem has reached the level of importance it has today. In view of the fact that the study of phase transitions, and the critical behavior of physical systems near these transitions, has always been one of the principal areas of research in statistical mechanics it is natural that a new class of techniques for studying field theories has grown up based on methods developed for application to problems of statistical mechanics. Notable in this respect is the Wilson renormalization group[3] (RG) or equivalently Kadanoff's block spin methods[4] which allow one to determine (in an almost imperial way) the large distance or bulk properties of a system from its local properties.

This feat is achieved by selectively focusing on the physics at each individual length scale between the microscopic level of the local interactions and the macroscopic scale where the bulk properties become manifest. This paper is intended as a pedagogical analysis of a simple field theoretical system which undergoes a phase transition by the methods of the RG. Along the way a transition will be made to a simple spin system analog. A phase

transition will be found for this spin system's quantum mechanical ground state which to the author's knowledge has not been previously reported.

In this discussion a starting simplification is made by putting the field theory on a lattice. This is done in what has come to be called the Hamiltonian formulation of a lattice theory wherein time remains a continuous parameter. This approach is equivalent to finding the properties of a "spin" system at zero temperature. What one desires to calculate are the ground state and low lying excitations (particles) of the system. A field theory Hamiltonian (H) on the lattice resembles a set of continuous spins which have, in general, complicated self interactions ($V(\phi)$) and ferromagnetic couplings (the spatial derivatives expressed as finite differences). By a change of basis from the natural Schrödinger representation in which the fields are diagonal to a basis of the oscillator states of the local terms in H one achieves an analogy to a quantized spin system where the spins take on discrete values, have ferromagnetic couplings, and interact with a "magnetic field" generated by the level structure of the local oscillators. This suggests a well-defined approximation scheme, albeit one which causes serious problems with Lorentz invariance. By restricting the number of levels which are allowed to participate in any oscillator one transforms the theory into one of interacting finite spins. An example of the reverse of this procedure is the Holstein-Primakoff substitution for spin waves. [5]

The resulting system may be studied by means of the RG to determine the global properties of the ground state as well as the low lying excitation

spectra. There are two areas of difficulty in carrying out this program in general. Both are related to the necessity of reducing the number of participating degrees of freedom as one changes the length scale. The procedure to be followed may be stated roughly as: (1) Divide the complete system into small subsystems. (2) Isolate the terms in the Hamiltonian which only deal with a particular substystem, and solve this finite problem exactly. (3) Re-express the sub-Hamiltonian in diagonal form in the new basis, and by the same transformation re-express the operators involving the degrees of freedom in this subsystem in the new basis. (4) Re-express the total Hamiltonian in terms of the subsystems. One has reduced the number of degrees of freedom in the problem but each of them has increased in complexity. This one overcomes, in the same spirit that we originally neglected the higher oscillator states, by neglecting all but the few lowest states of each subsystem. If one is dealing with a problem in more than one space dimension there is a further difficulty due to the fact that the number of local operators for each subsystem needed to describe the new Hamiltonian has also increased. Sometimes the solution to the first problem is also the solution to this problem, as all of the operators may become the same if very few states of the subsystem are kept. However, in general one must find some way of replacing these operators by some average or effective operators. Having dealt with the problems, or not, one may repeat the procedure again, (eventually, in fact, usually very soon one must face them), and again, and again until the quantities one is

interested in computing converge. What one discovers is fixed point behavior, eventually when one carries out this procedure nothing discernable happens. The new Hamiltonian is the same (up to possible rescaling of some quantities) as the last one, and one may quit. This is very reasonable when one realizes that, for example, in a three dimensional system where each iteration involves a change in the length scale of two, after 26 iterations each subsystem represents a mole of spins. When one has reached this point it is quite reasonable to suppose that the bulk properties have long since manifested themselves.

In Section II we will describe a procedure for replacing a lattice field theory, in this case $\lambda\phi^4$, by a system of coupled spins. In Sections III and IV we will investigate the resulting spin system by means of the renormalization group, discover that the system possesses a phase transition, and then perform some tests to see how well we have done in describing the system. In Section V it will be shown that the spin system has a well-defined continuum limit which is, however, essentially nonrelativistic in character. The implications of these results for the starting field theory will be discussed.

II. THE FINITE SPIN APPROXIMATION

The simplest field theory to exhibit a spontaneous breakdown of an internal symmetry is scalar ϕ^4 in one space dimension with a negative mass term. The Hamiltonian for this theory on a spatial lattice may be written

$$H = a\sum_i \left[\frac{\pi^{i\,2}}{2} + \lambda\,\phi^{i\,4} + \left(\frac{1}{a^2} + \mu^2\right)\phi^{i\,2} - \frac{1}{a^2}\,\phi^i\phi^i + 1\right] \tag{2.1}$$

where a is the lattice spacing $\pi^i = d\phi^i/dt$, and

$$\left[\pi^i,\ \phi^j\right] = -i\delta_{ij}/a \tag{2.2}$$

The spatial derivative has been approximated by finite differences.

Each of the local terms in H describes an anharmonic oscillator, and the nonlocal terms couple these oscillators. We can rewrite H in terms of the oscillator state at each site by solving the Schrödinger equation

$$\left(\frac{p^2}{2a^2} + \lambda x^4 + \left(\frac{1}{a^2} + \mu^2\right)x^2\right)|n\rangle = E_n|n\rangle \tag{2.3}$$

and computing the dipole matrix elements

$$\phi_{mn} = \langle m|x|n\rangle \tag{2.4}$$

This done, H can be written in the form

$$H = a \sum_i \left(E^i - \frac{1}{a^2} \phi^i \times \phi^{i+1} \right) \tag{2.5}$$

where E is a diagonal matrix, and ϕ is a matrix with non-vanishing elements between even and odd states. These matrices are infinite dimensional, and H acts in the basis generated from

$$\bigotimes_i | n_i > \quad 0 \leq n_i \leq \infty \tag{2.6}$$

What we propose as an approximation scheme is to consider a sequence of bases generated from

$$\bigotimes_i | n_i > \quad 0 \leq n_i \leq M \tag{2.7}$$

The Hamiltonian which results from this truncation describes a set of coupled spins with $2j + 1 = M + 1$.

The breakdown of the symmetry ($\phi \rightarrow -\phi$) in ϕ^4 is intimately related to the existence of two nearly degenerate oscillator states in the limit where μ^2 becomes large and negative. Here the level structure will resemble that depicted in Figure 1. We may expect that the qualitative features of this breakdown can be understood from a careful study of just these two levels. This is indeed the case and we will see that much of the conventional lore can be explained in terms of the physics of these two states.

The matrix ϕ has the form $\langle 0 | x | 1 \rangle \sigma_x$ in this approximation, and the two lowest entries of E can be written as $\frac{1}{2}(E_0 + E_1) + \frac{1}{2}(E_1 - E_0)\sigma_z$, where the σ's are the usual Pauli spinors. Thus H achieves the general form

$$H = a \sum \left(E^0 - \epsilon \sigma_z^i - g^2 \sigma_x^i \sigma_x^{i+1} \right) \tag{2.8}$$

with

$$\begin{aligned} E^0 &= \tfrac{1}{2}(E_0 + E_1) \\ \epsilon &= \tfrac{1}{2}(E_1 - E_0) \\ g^2 &= \frac{1}{a^2} |\langle 0 | x | 1 \rangle|^2 \end{aligned} \tag{2.9}$$

This is also the Hamiltonian of an Ising ferromagnet with a magnetic field perpindicular to the axis along which the spins couple, and is a problem of interest in its own right. It is also perhaps the simplest physical system to have nontrivial critical behavior at zero temperature. This analogy will be put to very good use in the sequel.

We would recover a great deal more of the structure of the theory if we kept the first four states in each oscillator, for then the theory could have bosonic excitations of one well or the other. This is not essential, though, to understand the breakdown of symmetry, and has not been pursued at this time.

III. THE RENORMALIZATION GROUP APPROACH

We now turn our attention to finding the ground state of the Hamiltonian (2.8). Following Wilson[3] we will attempt to treat the problem by successive rediagonalization of the lowest states of a sequence of Hamiltonians which describe different length scales. The procedure is essentially that outlined in the introduction. If we keep only two levels at each stage the procedure may be performed analytically, and is useful to gain an understanding of how things work.

Consider two adjacent spins, say s_0, and s_1. The terms in H which only involve these spins will be called their sub-Hamiltonian, for these spins it is

$$H_{0,1} = 2E^0 - \epsilon(\sigma_z^0 + \sigma_z^1) - g^2 \sigma_x^0 \sigma_x^1 \qquad (3.1)$$

This may be written as a four by four matrix

$$H_{0,1} \begin{pmatrix} -2\epsilon & 0 & 0 & -g^2 \\ 0 & 0 & -g^2 & 0 \\ 0 & -g^2 & 0 & 0 \\ -g^2 & 0 & 0 & 2\epsilon \end{pmatrix} + 2E^0 \qquad (3.2)$$

corresponding to the four possible states ($|\uparrow\uparrow>$, $|\uparrow\downarrow>$, $|\downarrow\uparrow>$, $|\downarrow\downarrow>$). In this basis the operators $\sigma_x^0 \otimes 1\!\!1^1$, and $1\!\!1^0 \otimes \sigma_x^1$ take the form

$$\sigma^L \equiv \sigma_x^0 \otimes \mathbb{1}^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\sigma^R \equiv \mathbb{1}^0 \otimes \sigma_x^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad (3.3)$$

The sub-Hamiltonian $H_{0,1}$ is easily diagonalized, giving eigenvalues $2E^0 \pm \sqrt{g^4 + 4\epsilon^2}$, and $2E^0 \pm g^2$. If σ^L, and σ^R are re-expressed in the new basis they take the form

$$\sigma^L = \begin{pmatrix} 0 & A & -B & 0 \\ A & 0 & 0 & B \\ -B & 0 & 0 & A \\ 0 & B & A & 0 \end{pmatrix}$$

$$\sigma^R = \begin{pmatrix} 0 & A & B & 0 \\ A & 0 & 0 & B \\ B & 0 & 0 & -A \\ 0 & B & -A & 0 \end{pmatrix} \qquad (3.4)$$

where

$$A^2 = \tfrac{1}{2}\left(1 + g^2/\sqrt{\left(g^4 + 4\epsilon^2\right)}\right)$$

$$B^2 = \tfrac{1}{2}\left(1 - g^2/\sqrt{\left(g^4 + 4\epsilon^2\right)}\right) \qquad (3.5)$$

and the matrices are arranged so that $H_{0,1}$ has its eigenvalues in increasing order on the diagonal. Notice that if at this stage we elected to keep all four states that σ^L and σ^R would be different operators, and in subsequent manipulations it would be necessary to always keep track of the left-most and right-most spins of the block. Supposing that for the ground state only the lowest two states are important, and thus only keeping the first two by two submatrix of each operator we find that σ^L and σ^R are the same and are just in the form $A\,\sigma_x$. This part of $H_{0,1}$ can also be written in terms of a constant and the spinor σ_z. The σ's no longer act on the spins but rather on the two lowest states of $H_{0,1}$.

Suppose now that all of the spins are arranged in pairs, and that this procedure has been carried out for every pair. The full Hamiltonian can now be rewritten approximately in terms of the lowest two states of each pair as

$$H \cong H_1 \equiv a_1 \sum_i \left(E_1^{\,0} - \epsilon_1 \sigma_z^{\,i} - g_1^{\,2} \sigma_x^{\,i} \sigma_x^{\,i+1}\right) \qquad (3.6)$$

Now the sum extends over all pairs. H_1 is in a form identical to $H \equiv H_0$ except that we have made the replacements

$$
\begin{aligned}
a_1 &= 2a \\
E_1^0 &= E_0^0 - \left(g_0^2 + \sqrt{g_0^4 + 4\epsilon_0^2}\right)/4 \\
\epsilon_1 &= \left(\sqrt{g_0^4 + 4\epsilon_0^2} - g_0^2\right)/4 \\
g_1^2 &= g_0^2\left(1 + g_0^2/\sqrt{g_0^4 + 4\epsilon_0^2}\right)/4
\end{aligned}
\tag{3.7}
$$

where we have designated the starting values of E^0, g^2, and ϵ with the subscript zero.

If the approximation has not been too severe, H_1 should describe the same large distance theory as H_0. The short distance properties of H_0 are replaced in H_1 by a contribution to the background energy density E^0. The parameters ϵ_1 and g_1^2 still describe the spectrum of excitations away from the ground state, which depends on the original length scale such that in terms of the pairs they are very roughly about half their original size. Having performed the analysis for the first iteration, the rest is easy. Simply carry out the replacements of Eq. (3.7) over and over again. E^0 will converge to a constant which can be interpreted as the ground state energy density per spin. The remaining part of H_n will still describe the low lying excitations. The most remarkable feature of

this iteration is fixed point behavior. Clearly if either ϵ or g^2 is zero, then they are both, except to be divided by two, reproduced unchanged. So these values of the parameters correspond to fixed points of the iteration. There is one other fixed point of Eq. (3.7) which is not so easy to see, but which occurs when $\epsilon/g^2 = 1.26$. The behavior of the parameters is easier to follow by considering what happens to their ratio. Defining $\alpha = 2\epsilon/g^2$ we find that under one iteration

$$\alpha_n \to \alpha_{n+1} = 2\sqrt{1+\alpha_n^2}\,\frac{\sqrt{1+\alpha_n^2}-1}{\sqrt{1+\alpha_n^2}+1} \equiv f(\alpha n) \qquad (3.8)$$

Fixed points in the iteration correspond to the solutions of $\alpha = f(\alpha)$, which are $\alpha = 0$, $\alpha \cong 2.52 \equiv \alpha_c$, and $\alpha = \infty$. If $0 < \alpha < \alpha_c$ then $f(\alpha) < \alpha$ so that in subsequent iterations α will decrease and approach zero where it must stop since $f(0) = 0$. If $\alpha > \alpha_c$ then $f(\alpha) > \alpha$ so that α increases indefinitely. $\epsilon = 0$ ($\alpha = 0$) and $g^2 = 0$ ($\alpha = \infty$) are then infrared stable fixed points while $\epsilon/g^2 = 1.26$ ($\alpha = \alpha_c$) is an ultraviolet stable fixed point. In Figure 2 the function $\alpha - f(\alpha)$ is plotted, and the behavior of α under successive iterations is indicated graphically. The implications of this kind of behavior are profound. The large distance or macroscopic properties of a system with a starting value of $\alpha < \alpha_c$ or $\alpha > \alpha_c$ are completely different. The system has a phase transition!

In both limits, after many iterations the Hamiltonian achieves a trivial form. If $\alpha_0 < \alpha_c$, the large distance theory is one described by $2^n \epsilon_n \to 0$, $2^n g_n^2 \to g_\infty^2$ i.e. a pure Ising system with no magnetic field. The ground state is doubly degenerate with all the spins polarized along $\pm \hat{x}$. The lowest energy excitation is the formation of one domain boundary with all the spins oppositely polarized to the left and right of the boundary. The energy of this state above the ground state is $2ag_\infty^2$. These excitations are the soliton or kink states of the broken ϕ^4 theory. If $\alpha_0 > \alpha_c$ the large distance theory is described by $2^n g_n^2 \to 0$, $2^n \epsilon_n \to \epsilon_\infty$.

This is a system of uncoupled block spins, each of which interacts with a magnetic field which is screened by the short distance fluctuations. The ground state is not degenerate and has all of the spins aligned along $\hat{z}$. The lowest energy excitation is a single spin flip with an energy $2a\epsilon_\infty$. These are the analogs of the normal bosonic states of unbroken ϕ^4. In both cases if α begins very near the critical value the energy of the lowest excitation becomes very small vanishing when $\alpha = \alpha_c$. This is another hallmark of a phase transition, the appearance of a massless excitation or infinite correlation length at the critical point.

We have made severe approximations in getting these results but have gained some insight into the structure of the theory at large distance. One now faces the task of investigating the validity of these approximations. Especially if one wishes to obtain useful quantitative results, and also to

check whether the qualitative features change with a more careful analysis. We can systematically improve the calculations of the spin system by keeping more states at each iteration. If the properties converge, and do not change in any qualitative way then we may feel that we understand the physics of this system. To study the iteration with more than two states requires the use of a computer, and a study of this kind has been carried out keeping 4 and 8 states requiring the diagonalization at each iteration of a 16×16 or a 64×64 matrix respectively. Keeping more states would have exceeded the available storage capacity of the computer. For four and eight states the procedure is very much like that already described for two states. For four states the starting Hamiltonian was the one generated in the study of the two state approximation, only this time all four states were kept. The left hand spin σ^L, and the right hand spin σ^R, are different operators and both of them are kept. The first iteration consists of solving the 16×16 sub-Hamiltonian

$$\epsilon^0 \otimes 1\!\!1^1 + 1\!\!1^0 \otimes \epsilon^1 - g^2 \sigma_R{}^0 \otimes \sigma_L{}^1 \tag{3.9}$$

where now the indices refer to blocks of two of the original spins and ϵ is the four by four matrix of Eq. (3.2). Diagonalizing this sub-Hamiltonian, then transforming $\sigma^{L0} \otimes 1\!\!1^1$ and $1\!\!1^0 \otimes \sigma^{R1}$ into the new basis, and then keeping only the first four levels completes the process. This iteration is now carried out until it has converged to a fixed point. The eight

state calculation is identical except that the starting Hamiltonian is generated from a block of three spins, after which one iterates by combining blocks in pairs. (There is another possible iteration scheme suggested here which would be to keep only four of the states generated by the block of three spins, and then in subsequent iterations combine three blocks at a time rather than two. An interesting question is which of these two approaches generally leads to a better answer. There is a two parameter approximation scheme, like Padé, i.e. combine n blocks at a time and keep M levels each time diagonalizing M^n states.)

The results of the numerical studies are quite similar to the two state case already studied. For $\alpha > \alpha_c$, or g_o^2 small one finds that $g_n^2 \to 0$ while $2^n\epsilon$ tends to some finite limit in the form

$$2^n \epsilon_n \to \frac{\epsilon_\infty}{2} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ or } \frac{\epsilon_\infty}{4} \begin{pmatrix} -7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (3.10)$$

There is a non-degenerate ground state, and the remaining states are all degenerate and split up by some amount which depends on the starting values of ϵ and g^2. This is an attempt by the theory to represent the first few states of a continuum of single particle excitations labeled by a momentum. The static energy of this excitation is the gap between the ground state and the first excited state, $2a\,\epsilon_\infty$. As α_0 is increased towards

a critical value this energy difference vanishes. When α_0 increases above this critical value ($\epsilon/g^2 \cong .97$ for 4 states and $\epsilon/g^2 \cong .98$ for eight states) things change dramatically. $2^n g_n^2$ remains finite, also the splitting matrix ϵ remains finite (unlike the two state case), but the character of both matrices changes. A typical final form for the four state case would be

$$2^n \epsilon_n \to \epsilon_\infty \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.11}$$

$$\sigma^L \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \sigma^R \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The resulting Hamiltonian is strictly diagonal in spin space. The ground state is two-fold degenerate where the two possible ground states are to have all of the block spins in either the first (0) or second (1) level. Of all of the possible excitations the two candidates for lowest energy are

a) ....000011111... or ....1111100000...
$E = 2ag_\infty^2$ (3.12)

b) ....0000311111... or ....111120000...
$E = 2a\epsilon_\infty$

Both of these correspond to the formation of domains, or soliton like states. In the calculations b) was always found to have the smallest energy. (In the mean field approximation to be discussed later it is also found that the lowest energy state naively has some structure at the domain boundary). The results for the energy of the lowest excitation above the ground state and the position of the phase transition are indicated in Figure 3. The starting values were $E_0^0 = 0$, $\epsilon_0 = 1$. It is pleasing to note that the qualitative features do not change as the approximation is improved, and that the quantitative details seem to be rapidly converging. We will return to more detailed results of these calculations in the following sections, but for reference they are collected in Table I.

IV. SIMPLE MINDED CHECKS

The renormalization group calculations by their nature provide upper bounds on the energy density in the ground state. This is because the calculation is really a variational calculation in a very carefully chosen basis. We can get an estimate of how accurately we have treated the ground state by comparing the RG with other upper and lower bounds on the energy density. It is also instructive to compare the properties of the phase transition with mean field results.

The mean field or Hartree approximation results if we make the ansatz for the ground state

$$|0> = \bigotimes_i \chi_\theta^i \qquad \chi_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{4.1}$$

The resulting ground state energy density is an upper bound to the exact result, and is given by

$$E^0 - \epsilon \cos 2\theta - g^2 \sin^2 2\theta \tag{4.2}$$

This is minimized by setting $\theta = 0$ when $\epsilon/g^2 > 2$, and when $\epsilon/g^2 \leq 2$ by choosing θ such that

$$\cos 2\theta = \epsilon/2g^2 \tag{4.3}$$

This has the earmarks of a second order phase transition. The energy density has continuous first, but not second derivative. The matrix

element of σ_x acts as the order parameter, and is zero for $\epsilon \geq 2g^2$, but for $\epsilon < 2g^2$ is given by $\sqrt{1 - \epsilon^2/4g^4}$ which has the form $(\alpha_c - \alpha)^{\frac{1}{2}}$ near the phase transition, characteristic of the mean field approximation. In contrast the RG calculations for 2, 4, and 8 states kept give the form $(\alpha_c - \alpha)^p$ where $p \cong 0.40$, 0.46, and 0.45 respectively. The critical value of ϵ/g^2 is 2.0 here compared to 1.26, 0.97, and 0.98 for the RG calculation.

In the limits $\epsilon/g^2 >> 1$, and $\epsilon/g^2 << 1$ we can use simple perturbation theory to compute the low lying spectra above the Hartree ground state. For $g^2/\epsilon << 1$ the ground state has all the spins polarized along $\hat{z}$. The first excited state of the unperturbed Hamiltonian is degenerate and corresponds to flipping one spin anywhere. Diagonalizing H in these states gives an energy above the ground state for a spin wave of wave vector k

$$\epsilon(k) = 2a\left(\epsilon - g^2 \cos(ka)\right) \tag{4.4}$$

where $-\pi/a \leq k \leq \pi/a$. The lowest excitation, k = 0, has an energy which agrees very well with the RG results when g^2/ϵ is small. For $\epsilon/g^2 << 1$ the ground state is degenerate between the choices $\pm\theta$ in Eq. (4.3). First leaving θ as determined there, one can do a variational calculation for a static domain wall using a state which has all of the spins in $\chi_{\pm\theta}$ to the left or right of the domain center except leaving a few spins in the center free, and varying them to minimize the energy. If just one spin is free it will orient along $\hat{z}$ giving an energy

$$\epsilon(0) = a\epsilon\left(\cos(2\theta) - 1\right) + 2ag^2\sin^2(2\theta) \qquad (4.5)$$

which is a little smaller than the energy of an abrupt break from $-\theta$ to θ, $2ag^2\sin^2 2\theta$. If several spins are left free, a smooth transition is formed which sharpens up as $\epsilon \to 0$. This can be compared with the behavior in the RG calculation where some structure was also preferred at the domain boundary rather than an abrupt break. A second exercise is to set $\epsilon = 0$ and find the states of the unperturbed Hamiltonian. Then treating ϵ in the lowest order perturbation theory gives the analog of Eq. (4.4)

$$\epsilon(k) = 2a\left(g^2 - \epsilon\cos(ka)\right) \qquad (4.6)$$

Again the lowest excitation agrees well with the RG results when ϵ/g^2 is small. A comparison of Eqs. (4.4, 6) shows a symmetry under the exchange $g^2 \leftrightarrow \epsilon$ for the perturbation theory results. It is very tempting to speculate that this is a property of the exact theory in which case the phase transition would be forced to occur precisely at $\epsilon/g^2 = 1$. The best approximate value of 0.98 for ϵ/g^2 is also suggestive. One of the most remarkable things is just how well lowest order perturbation theory works for the static energy of the lowest excitations. In Figure 4 the 8 state RG results are compared to Eqs. (4.4) and (4.6).

Another check is obtained from a lower bound on the energy density, which may be gotten by solving a system of N spins exactly with the replacement $g^2 \to g^2N/(N-1)$[6]. The ground state energy divided by N

is a lower bound on the exact energy density. Results of all the calculations of the energy density for $E_0^0 = 0$ and $\epsilon_0 = 1$ are presented as a function of g_0 in Figure 5. The positions of the approximate phase transitions are indicated for the upper bounds. The lower bound derives from a finite system and cannot have a phase transition. This fact yields an interesting byproduct for we can see in Figure 6 how the level structure of the finite system already begins to show clearly the form of the full theory as the number of sites is increased from two to six. The first two excited states remain nearly degenerate for $\epsilon/g^2 > 1$, and then the first state comes down and joins the ground state while the second state goes up rapidly.

V. TOWARDS THE CONTINUUM

If we adopt the point of view that the small momentum or large distance behavior of the $\lambda\phi^4$ theory which we began to study in Section II is insensitive to the value of the cutoff (1/a) then we are through having shown that the theory possesses a phase transition with all of the properties one expected. The qualitative features of this transition are determined solely in terms of the physics of the two lowest levels. The finer details, like whether the excitations in the broken phase are solitons or kinks (domains) or normal bosonic excitations of one well of the potential, depend on the higher states of the local oscillators in H which we have not considered. It is still interesting to see how far one can go in constructing a continuum theory from the lattice theory, by sending the lattice spacing to zero holding the physics of the low lying excitations fixed. In particular we first have to find out whether it is even possible to make a finite theory as the lattice spacing becomes small. This is a very simple theory and there are only two things we can demand. The first is that the mass of the lowest excitation should remain finite at some value we will call m_R for the renormalized mass. The second is that the relativistic energy-momentum relation

$$\epsilon(k) = m_R + \frac{k^2}{2m_R} + O(k^4) \tag{5.1}$$

should hold when k is small. These two conditions completely fix the two

ϵ and g^2 for any value of the lattice spacing (or inverse cutoff). At this point the extent to which we have butchered the starting theory is very obvious. In keeping only two states of the original local oscillators we had assumed that the remaining levels were well split up, and could be ignored. A consequence of this assumption is that we have forever lost one of our parameters in the starting theory. To begin we will take a less ambitious task and set the following exercise: suppose that $m_R = 1$ GeV, and thinking that this is a reasonable approximation we ask that there be 20 lattice points in one Compton wavelength or that $a = 10^{-15}$ cm $= 0.05\ \text{GeV}^{-1}$. First we will try to use perturbation theory to find out what ϵ and g^2 must be, supposing that g^2/ϵ is small. For small g^2 and small k from Eq. (4.4) we have

$$\epsilon(k) \cong 2a\left(\epsilon_0 - g_0^2\right) + g_0^2 a^3 k^2 \tag{5.2}$$

so we must have

$$m_R = 2a\left(\epsilon_0 - g_0^2\right) = 1/\left(2g_0^2 a^3\right) \tag{5.3}$$

For our problem this requires

$$g_0^2 = 4000\ \text{GeV}^2 \qquad \epsilon_0 = 4010\ \text{GeV}^2 \qquad \frac{g_0^2}{\epsilon_0} = 0.997 \tag{5.4}$$

We have <u>not</u> satisfied our starting assumption that g^2/ϵ was small. Now a knowledge of the RG transformation is useful. In the simplest recursion

where only two states were kept by means of n RG transformations one could rewrite H as

$$H_n = 2^n a \sum_i \left(E_n^0 - \epsilon_n \sigma_3^i - g_n^2 \sigma_x^i \sigma_x^{i+1} \right) \tag{5.5}$$

The only thing different here for our purposes is the length scale. a has been replaced by $2^n a$. Now we should have

$$m_R = 2^{n+1} a \left(\epsilon_n - g_n^2 \right) = 1/\left(2^{3n+1} g_n^2 a^3 \right) \tag{5.6}$$

If n is chosen as 6 we find that

$$g_6^2 = 0.015 \text{ GeV}^2 \qquad \epsilon_6 = 0.640 \text{ GeV}^2 \qquad \frac{g_6^2}{\epsilon_6} = 0.024 \tag{5.7}$$

and one can feel that perturbation theory is a reasonable approach. Now at last the RG is put to use to find out what starting values ϵ_0 and g_0^2 give the required values after 6 iterations. We actually have a pretty good idea what this value will be since we know that there is an ultraviolet stable fixed point and 6 iterations would take one very close to it. Having solved our exercise we may become more bold and set $a = 10^{-20}$ cm. The same procedure will again yield a solution except that this time more iterations will be required before one may safely apply perturbation theory. In the limit infinite numbers of iterations will be required and we will be forced to have g_0^2/ϵ_0 very close to the critical value with corrections which are very small, crudely something like the lattice spacing away.

Just how far away depends on the critical exponents which describe the phase transition. Before we consider this interesting point, we are obliged to note that we could have initially assumed that ϵ was going to be small compared to g^2. If ϵ can be considered a perturbation then

$$\epsilon(k) \cong 2a\left(g^2 - \epsilon \cos(ka)\right) \tag{5.8}$$

Precisely the same analysis would then yield

$$\epsilon_0 = 4000\ \text{GeV}^2 \qquad g_0^2 = 4010\ \text{GeV}^2 \qquad \frac{\epsilon_0}{g_0^2} = 0.997 \tag{5.9}$$

and again 6 iterations of the RG would give $\epsilon_6/g_6^2 = 0.024$. This is a remarkable result when $a \to 0$ the same choice of parameters differing only by an infinitesimal amount can lead to two completely different finite theories, and for no other choice can one obtain a finite theory. Actually this is not surprising at all if one studies the form of Eq. (2.1) one realizes that it is μ^2 not $\left(\mu^2 + \frac{1}{a^2}\right)$ which we expect to be important when $a \to 0$, but if we had begun just with the spin system we might have been surprised.